Midterm Exam Calculus 2

19 March 2018, 9:00-11:00

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. **[5+10+5=20 Points]** Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$
   (a) Show that f is not continuous at $(x,y) = (0,0)$. [Hint: consider f along the curve parametrized as $(x(t), y(t)) = (t, t^2)$.]
   (b) Show that the directional derivatives of f at $(x,y) = (0,0)$ exist in all directions $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ with $v^2 + w^2 = 1$ by using the definition of directional derivatives. Note that you will have to distinguish between the two cases $w = 0$ and $w \neq 0$.
   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. **[15+10=25 Points]** Let $a, b > 0$. Consider the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$ with
$$\mathbf{r}(t) = e^{at}\cos(bt)\,\mathbf{i} + e^{at}\sin(bt)\,\mathbf{j} + e^{at}\,\mathbf{k}.$$
   (a) Determine the length of the curve and its parametrization by arclength.
   (b) At each point on the curve, determine the curvature of the curve.

3. **[5+10+10=25 Points]** Consider the ellipsoid
$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$
which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.
   (a) Determine the tangent plane of the ellipsoid at the point (x_0, y_0, z_0).
   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto z$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).
   (c) Use the method of Lagrange multipliers to find the points closest to and farthest away from the origin.

4. **[20 Points]** Evaluate the triple integral
$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy$$
by using cylinder coordinates.

1.

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) for $(x(t), y(t)) = (t, t^2)$,

$$f(x(t), y(t)) = \begin{cases} \dfrac{t^2 t^2}{t^4+t^4} & , t \neq 0 \\ 0 & , t = 0 \end{cases} = \begin{cases} \frac{1}{2} & , t \neq 0 \\ 0 & , t = 0 \end{cases}$$

$\Rightarrow$ f has no limit at $(x,y) = (0,0)$

(b) let $u = (v,w) \in \mathbb{R}^2$, $v^2+w^2 = 1$

$$D_u f(0,0) = \lim_{h \to 0} \frac{f(hu) - f(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3 v^2 w}{h^4 v^4 + h^2 w^2} - 0}{h}$$

for $w = 0$,

$$D_u f(0,0) = \lim_{h \to 0} \frac{\frac{0}{h^4 v^4} - 0}{h} = 0$$

for $w \neq 0$,

$$D_u f(0,0) = \lim_{h \to 0} \frac{h^4 v^2 w}{h^4 v^4 + h^2 w^2} = \lim_{h \to 0} \frac{h^2 v^2 w}{h^2 v^4 + w^2} = 0$$

as $\lim\limits_{h \to 0} h^2 v^2 w = 0$

and $\lim\limits_{h \to 0} h^2 v^4 + w^2 = w^2 \neq 0$

c) f is not differentiable at $(x,y) = (0,0)$ as f is not continuous at $(x,y) = (0,0)$.

2.

a) $r(t) = e^{at}\cos(bt)\,i + e^{at}\sin(bt)\,j + e^{at}\,k.$

$$\Rightarrow r'(t) = \left[a e^{at}\cos(bt) - b e^{at}\sin(bt)\right] i$$
$$+ \left[a e^{at}\sin(bt) + b e^{at}\cos(bt)\right] j$$
$$+ a e^{at} k$$

$$\|r'(t)\| = e^{at}\left[\left(a\cos(bt) - b\sin(bt)\right)^2 + \left(a\sin(bt) + b\cos(bt)\right)^2 + a^2\right]^{1/2}$$
$$= e^{at}\left[2a^2 + b^2\right]^{1/2}$$

$$\Rightarrow s(t) = \int_0^t \|r'(\tau)\|\,d\tau = \left[2a^2+b^2\right]^{1/2} \cdot \int_0^t e^{a\tau}\,d\tau = \frac{1}{a}\left[2a^2+b^2\right]^{1/2}\left(e^{at}-1\right)$$

$$\Rightarrow S(1) = \frac{1}{a}\left[2a^2+b^2\right]^{1/2}\left(e-1\right) = L \text{ (length of the curve)}$$

solving for t gives:

$$t(s) = \frac{1}{a}\ln\left[\frac{as}{\left[2a^2+b^2\right]^{1/2}} + 1\right], \quad s \in [0, L]$$

$$\tilde{r}(s) = r(t(s)) = \left[\frac{as}{\left[2a^2+b^2\right]^{1/2}} + 1\right]\left(\cos\left(b\,t(s)\right) i + \sin\left(b\,t(s)\right) j + k\right)$$

b) to determine curvature first determine unit tangent vector at $r(t)$:

$$T = \frac{1}{\|r'(t)\|}\,r'(t) = \frac{e^{-at}}{\left[2a^2+b^2\right]^{1/2}}\Big[\left(a e^{at}\cos(bt) - b e^{at}\sin(bt)\right) i$$
$$+ \left(a e^{at}\sin(bt) + b e^{at}\cos(bt)\right) j$$
$$+ a e^{at} k\Big]$$

$$= \frac{1}{\left[2a^2+b^2\right]^{1/2}}\Big[\left(a\cos(bt) - b\sin(bt)\right) i$$
$$+ \left(a\sin(bt) + b\cos(bt)\right) j$$
$$+ a k\Big]$$

$$\frac{d}{dt} T = \frac{1}{\left[2a^2+b^2\right]^{1/2}} \Big[\left(-ab\sin(bt) + b^2 \cos(bt)\right) i$$
$$+ \left(ab\cos(bt) - b^2 \sin(bt)\right) j \Big]$$

$$\left\| \frac{dT}{dt} \right\| = \frac{1}{\left[2a^2+b^2\right]^{1/2}} \left[a^2b^2 + b^4\right]^{1/2}$$

curvature:

$$k = \left\| \frac{dT}{dt} \right\| \frac{1}{\|r'(t)\|} = \frac{\left[a^2b^2+b^4\right]^{1/2}}{\left[2a^2+b^2\right]^{1/2}} \quad e^{-at}\left[2a^2+b\right]^{-1/2}$$

$$= \frac{\left[a^2b^2+b^4\right]^{1/2}}{2a^2+b^2}$$

3. a) Set $F(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$

$\Rightarrow x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \Leftrightarrow F(x,y,z) = 3$

Then $\nabla F(x_0,y_0,z_0)$ is perpendicular to the ellipsoid at (x_0,y_0,z_0).

As $\nabla F(x,y,z) = 2x\,i + \frac{1}{2}y\,j + \frac{2}{9}z\,k$

we get $\nabla F(x_0,y_0,z_0) = \nabla F(1,2,3) = 2i + 1j + \frac{2}{3}k$

$\Rightarrow$ tangent plane is given by the equation

$$\nabla F(x_0,y_0,z_0)\,(r - r_0) = 0$$

where $r = xi + yj + zk$

$r_0 = x_0 i + y_0 j + z_0 k$

which gives

$$\left(2i + 1j + \frac{2}{3}k\right) \cdot \left((x-1)i + (y-2)j + (z-3)k\right) = 0$$

$$\Leftrightarrow (2x-2) + (y-2) + \frac{2}{3}(z-3) = 0$$

$$\Leftrightarrow 2x + y + \frac{2}{3}z = 6$$

b) For F as defined in part (a) we have

$$\frac{\partial}{\partial z} F(x,y,z) = \frac{2}{9} z \text{ which gives } \frac{\partial}{\partial z} F(x_0,y_0,z_0) = \frac{2}{9} z_0 = \frac{2}{3}.$$

As $\frac{\partial}{\partial z} F(x_0,y_0,z_0) \neq 0$ there exist by the Implicit Function Theorem a neighborhood U of (x_0,y_0) in $\mathbb{R}^2$, a neighb. V of z_0 in $\mathbb{R}$ and a function $f: U \to V$ such that if $F(x,y,z) = 3$ for $(x,y) \in U$ and $z \in V$ then $z = f(x,y)$. Both F and f are C^1 functions

$$f_x(x_0,y_0) = -\left.\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=(x_0,y_0,z_0)} = -\frac{2x_0}{\frac{2}{9} z_0} = -9\frac{x_0}{z_0} = -3$$

$$f_y(x_0,y_0) = -\left.\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right|_{(x,y,z)=(x_0,y_0,z_0)} = -\frac{\frac{1}{2}y_0}{\frac{2}{9} z_0} = -\frac{9}{4}\frac{2}{3} = -\frac{3}{2}$$

Linearization of f at (x_0,y_0):

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 3 - 3(x-1) - \frac{3}{2}(y-2)$$

The graph of L satisfies

$$z = L(x,y) \iff z = 3 - 3(x-1) - \frac{3}{2}(y-2)$$
$$\iff z = 3 - 3x + 3 - \frac{3}{2}y + 3$$
$$\iff z = 9 - 3x - \frac{3}{2}y$$
$$\iff \frac{2}{3}z + 2x + y = 6 \quad \text{(agrees with (a))}$$

3. c) distance square of the point (x,y,z) to the origin is given by $g(x,y,z) = x^2+y^2+z^2$.

Let F be defined as the part (a). Then at an extremum of g restricted to $F(x,y,z)=3$ there exist $\lambda \in \mathbb{R}$ such that $\nabla g(x,y,z) = \lambda \nabla F(x,y,z)$.

We have to solve the latter equation together with $F(x,y,z)=3$ for x,y,z and λ.

$$\left.\begin{array}{l} 2x = \lambda 2x \\ 2y = \lambda \frac{1}{2} y \\ 2z = \lambda \frac{2}{9} z \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x=0 \text{ or } \lambda=1 \\ y=0 \text{ or } \lambda=4 \\ z=0 \text{ or } \lambda=9 \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3 \end{array}\right.$$

$\Leftrightarrow$ $x=0,\ y=0,\ \lambda=9$

or

$x=0,\ z=0,\ \lambda=4$ with $x^2+\frac{y^2}{4}+\frac{z^2}{9}=3$

or

$y=0,\ z=0,\ \lambda=1$

$\Leftrightarrow$ $z^2=27,\ \lambda=9$

or

$y^2=12,\ \lambda=4$

or

$x^2=3,\ \lambda=1$.

Filling in the resulting points into g gives

$g(0,0,\pm\sqrt{27}) = 27$

$g(0,\pm\sqrt{12},0) = 12$

$g(\pm\sqrt{3},0,0) = 3$

$\Rightarrow$ at $(x,y,z)=(0,0,\pm\sqrt{27})$ points on the ellipsoid are furthest away and at $(x,y,z)=(\pm\sqrt{3},0,0)$ points on the ellipsoid are closest to the origin

4.

Evaluate $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{0}^{4-x^2-y^2} e^{x^2+y^2+z}\, dz\, dx\, dy$$

by using cylinder coordinates.

Cylinder coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

The integration with respect to x and y is over a disk of radius 1. The integral in cylinder coordinates hence is

$$\int_0^{2\pi}\int_0^1\int_0^{4-r^2} e^{r^2} e^{z} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r e^{r^2}\left(e^{4-r^2} - e^{0}\right) dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r e^{4} - r e^{r^2}\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} r^2 e^4 - \frac{1}{2} e^{r^2}\right)\Bigg|_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} e^4 - \frac{1}{2} e^1 - \left(0 - \frac{1}{2} e^0\right)\right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} e^4 - e^1 + 1\right) d\theta$$

$$= \pi\left(e^4 - e^1 + 1\right)$$